TOWARDS A CONJECTURE OF PAPPAS AND RAPOPORT ON A SCHEME ATTACHED TO THE SYMPLECTIC GROUP

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Abstract

Let $n = 2r$ be an even integer. We consider a closed subscheme V of the scheme of $n \times n$ skew-symmetric matrices, on which there is a natural action of the symplectic group $Sp(n)$. Over a field F with char $F \neq 2$, the scheme V is isomorphic to the scheme appearing in a conjecture by Pappas and Rapoport on local models of unitary Shimura varieties. With the additional assumption char $F = 0$ or char $F > r$, we prove the coordinate ring of V has a basis consisting of products of pfaffians labelled by King's symplectic standard tableaux with no odd-sized rows. When n is a multiple of 4, the basis can be used to show that the coordinate ring of V is an integral domain, and this proves a special case of the conjecture by Pappas and Rapoport.

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Contents

1 Introduction

Let F be an infinite field and n be a positive integer. It is well known that for each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of some positive integer with $0 < \lambda_k \leq \cdots \leq \lambda_1 \leq n$, there is an associated $GL(n, F)$-module such that the bideterminants labelled by the standard tableaux of shape λ form a basis of the module. This module is called the Weyl module and is irreducible when F is of characteristic zero.

Assume n is an even integer. For the symplectic case, Berele [1] has given a basis of the irreducible $Sp(n, F)$-module, called the symplectic Weyl module, over a field F of characteristic zero where the basis is labelled by the symplectic standard tableaux (defined by King [8]; see Definition 4.1). In [5], Donkin proves this result for arbitrary infinite field. Donkin shows by a symplectic version of the Carter-Lusztig Lemma that the bideterminants indexed by the symplectic standard tableaux of shape λ form a basis of the $Sp(n, F)$-module, which is defined as the span of all bideterminants associated to the tableaux of shape λ. (This module need not be irreducible anymore.)

In this thesis, we are interested in a scheme defined as follows: Let R be a commutative ring with unity and let $n = 2r$ be an even integer. Let V be the scheme of $n \times n$ matrices $Y = (Y_{ij})$ over Spec R such that

$$Y = -Y^T, \quad Y_{ii} = 0 \text{ for } i = 1, \ldots, n, \quad Y^T J Y = 0, \quad \text{and} \quad \mathrm{char}_{(-JY)}(T) = T^n, \tag{1.1}$$

where J is the $n \times n$ matrix

$$J = \left[\begin{array}{ccc|ccc} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ \hline -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \end{array}\right]. \tag{1.2}$$

The first two conditions in (1.1) imply that V is a closed subscheme of the scheme of $n \times n$

skew-symmetric matrices over Spec R. (The second condition is redundant if $2 \in R^{\times}$.) If a tableau has no odd-sized rows, we call it an *even-tableau* (and *even-tableaux* for the plural form). In [4, Ch.6], De Concini and Procesi show that there is an R-basis of the ring $R[Y_{ij}]/(Y + Y^T, Y_{11}, \dots, Y_{nn})$ indexed by the standard even-tableaux, where each such tableau corresponds to a product of pfaffians. On the other hand, there is a natural action of the symplectic group $Sp(n) := \{g \mid g^T J g = J\}$ on V by $Y \cdot g = g^T Y g$. One aim of this thesis is to find an R-module basis for the coordinate ring of V, denoted by $R[V]$, in terms of tableaux when R is given some suitable conditions. We omit the case $n = 2$ since $R[V]$ is isomorphic to R in this case given that $2 \in R^{\times}$. In Section 4 we prove the following:

Theorem 1.1. Let $n = 2r$ be an even integer with $n \geq 4$. When the scheme V is defined over a field F with char $F = 0$ or char $F > r$, there is an F-basis for the coordinate ring of V consisting of products of pfaffians labelled by the symplectic standard even-tableaux.

In Section 3 we develop some relations between pfaffians for later use. In Section 4 we define the symplectic standard even-tableaux and show that they can be used to label a basis of the coordinate ring of V. In Section 5 we prove a special case of Pappas and Rapoport's conjecture [10, Conj.5.2] when n and F meet some additional conditions.

Conjecture 1.2. (Pappas and Rapoport, 2009) Let F be a field with char $F \neq 2$ and let n be an integer divisible by 4. Let W be the scheme of $n \times n$ matrices X over Spec F with

$$-JX^T J = X, \quad X^2 = 0, \quad \text{and} \quad \text{char}_X(T) = T^n. \tag{1.3}$$

Then W is reduced. [1]

Assume that n is divisible by 4. Taking $R = F$, there is an isomorphism $W \to V$ given by $X \mapsto JX$. When F is a field with char $F = 0$ or char $F > r$, we prove that the coordinate ring of V is in fact an integral domain (Theorem 5.3) so the same holds true for W as well.

Theorem 1.3. Let n be a multiple of 4 and F be a field with char $F = 0$ or char $F > r$. Let W be the scheme defined as in Conjecture 1.2. Then the coordinate ring of W is an integral domain.

[1] In fact, Pappas and Rapoport formulate a more general conjecture for any even n, depending on a partition of n into two even parts; the version we have stated is the case $r = s$ in their notation.

2 Background and motivation

In this section we sketch the background of Conjecture 1.2 and our motivation. This section may be skipped by readers since it does not affect what comes in the subsequent sections.

We are interested in finding moduli descriptions of local models of Shimura varieties. A PEL type Shimura variety is a moduli space of polarized abelian varieties endowed with endomorphisms and level structure. When a Shimura variety is of PEL type, there are versions of the moduli problem defining models over the ring of integers in the completion of the reflex field at some place over a prime p. One seeks to determine whether these models are flat and to understand their singularities.

In general a Shimura variety is attached to a Shimura datum, which is a triple $(\mathbf{G}, \mathbf{K}, X)$ consisting of a reductive algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, a compact open subgroup $\mathbf{K}$ of the finite-adelic points $\mathbf{G}(\mathbb{A}_f)$ of $\mathbf{G}$, and a $\mathbf{G}(\mathbb{R})$-conjugacy class X of homomorphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathbf{G}_\mathbb{R}$. Fix a prime number p and assume that the Shimura datum is of PEL type, that $\mathbf{K}$ factorizes as

$$\mathbf{K} = K_p \cdot K^p \subset \mathbf{G}(\mathbb{A}_f) = \mathbf{G}(\mathbb{Q}_p) \times \mathbf{G}(\mathbb{A}_f^p),$$

and that K_p is the stabilizer of a self-dual periodic lattice chain in the defining PEL data. We define $\mathcal{G}$ as the canonical smooth group scheme over $\mathbb{Z}_p$ which has generic fiber $\mathbf{G}$ and connected special fiber with $\mathbf{K} = \mathcal{G}(\mathbb{Z}_p)$. In this case natural integral models $\mathcal{S}^{naive}$ (named *naive models*) are defined in [13] over $\mathcal{O}_{\mathbf{E}_\wp}$, the ring of integers in the $\wp$-adic complection $\mathbf{E}_\wp$, where $\wp$ is a place of the reflex field $\mathbf{E}$ over a prime p. When $\mathbf{G}$ splits over an unramified extension of $\mathbb{Q}_p$ and the localized group $\mathbf{G}_{\mathbb{Q}_p} := \mathbf{G} \otimes_\mathbb{Q} \mathbb{Q}_p$ has simple factors of types only A and C, Görtz [6, 7] showed that these models are flat and have reduced special fibers and that their irreducible components are normal with rational singularities. However, in the case that $\mathbf{G}$ is a unitary similitude group which only splits over a ramified extension of $\mathbb{Q}_p$, it was shown in [9] that the naive models may fail to be flat. One way to refine the naive models is by imposing additional conditions on the moduli problem in order to get flat closed subschemes of the naive models. Under our assumptions above, these problems can be reduced to problems of the corresponding *naive local models* M^{naive}, and sometimes it leads to very explicit problems about schemes defined by matrix identities as we will see later in this section.

The general ideal is that we want to obtain the integral Shimura models $\mathcal{S}$ which have good properties like flatness from the *local models* M^{loc} via the cartesian diagram

$$\begin{tikzcd}
\mathcal{S} \arrow[r] \arrow[d] & \left[M^{loc}/\mathcal{G}_{\mathcal{O}_{\mathbf{E}_\wp}}\right] \arrow[d] \\
\mathcal{S}^{naive} \arrow[r] & \left[M^{naive}/\mathcal{G}_{\mathcal{O}_{\mathbf{E}_\wp}}\right].
\end{tikzcd}$$

More precisely, M^{loc} should be equipped with an action of $\mathcal{G}_{\mathcal{O}_{\mathbf{E}_\wp}}$ and a smooth morphism of algebraic stacks

$$\mathcal{S} \to \left[M^{loc}/\mathcal{G}_{\mathcal{O}_{\mathbf{E}_\wp}}\right]$$

of relative dimesion dim $\mathbf{G}_{\mathbb{Q}_p}$. Equivalently, we should have a local model diagram of $\mathcal{O}_{\mathbf{E}_\wp}$-schemes

$$\begin{tikzcd}
& \widetilde{\mathcal{S}} \arrow[dl, "\psi"'] \arrow[dr, "\widetilde{\phi}"] & \\
\mathcal{S} & & M^{loc}
\end{tikzcd}$$

in the sense of [13], where ψ is a $\mathcal{G}_{\mathcal{O}_{\mathbf{E}_\wp}}$-torsor and $\widetilde{\phi}$ is smooth and $\mathcal{G}_{\mathcal{O}_{\mathbf{E}_\wp}}$-equivariant. Then for every $x \in \mathcal{S}(\overline{\mathbb{F}}_p)$, there exists $\overline{x} \in M^{loc}(\overline{\mathbb{F}}_p)$ unique up to the action of $\mathcal{G}(\overline{\mathbb{F}}_p)$, such that there is an isomorphism between the strict henselization of $\mathcal{S}$ at x and that of M^{loc} at $\overline{x}$ [11, pp.140]. In this sense, M^{loc} are expected to model the singularities of the integral Shimura models $\mathcal{S}$.

Most notations and definitions in this section closely follow [10]. Let K be an imaginary quadratic field and assume that $K/\mathbb{Q}$ is ramified over p with $p \neq 2$. Let $F = K \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Then F is a discretely valued field and we denote by $\mathcal{O}_F$ its ring of integers, $\pi \in \mathcal{O}_F$ a uniformizer such that π^2 is a uniformizer in $\mathbb{Z}_p$, and $k = \mathcal{O}_F/\pi\mathcal{O}_F$ its residue field.

Let $n \geq 3$ and let V be a F-vector space of rank n with a basis $e_1, \ldots, e_n$. A *lattice chain* in V is a set of $\mathcal{O}_F$-lattices in V totally ordered under inclusion. A lattice chain $\mathcal{L}$ is called *periodic* if for every lattice Λ in $\mathcal{L}$ and $a \in F^\times$, the lattice $a\Lambda$ is also in $\mathcal{L}$. For $i = 0, \ldots, n-1$, define $\mathcal{O}_F$-lattices Λ_i as

$$\Lambda_i := \mathrm{span}_{\mathcal{O}_F}\{\pi^{-1}e_1, \ldots, \pi^{-1}e_i, e_{i+1}, \ldots, e_n\} \subset V.$$

By setting $\Lambda_{i+tn} = \pi^{-t}\Lambda_i$, we obtain a periodic lattice chain

$$\cdots \subset \Lambda_{-2} \subset \Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots,$$

which we call the *standard lattice chain*.

Let ϕ denote the $F/\mathbb{Q}_p$-Hermitian form on V such that

$$\phi(e_i, e_{n-j+1}) = \delta_{ij}$$

for every $i, j \in \{1, \ldots, n\}$. In other words, ϕ is represented by the following matrix

$$\begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

with respect to the basis $e_1, \ldots, e_n$. Note that there is an associated (alternating) $\mathbb{Q}_p$-bilinear form

$$\begin{aligned} \langle\ ,\ \rangle : V \times V &\longrightarrow \mathbb{Q}_p \\ (x, y) &\longmapsto \frac{1}{2}\,\mathrm{tr}_{F/\mathbb{Q}_p}(\pi^{-1}\phi(x, y)). \end{aligned}$$

For any $\mathcal{O}_F$-lattice Λ in V, its common $\langle\ ,\ \rangle$- and ϕ-dual is denoted by $\widehat{\Lambda}$, i.e.

$$\widehat{\Lambda} := \{x \in V \mid \langle \Lambda, x\rangle \subset \mathbb{Z}_p\} = \{x \in V \mid \phi(\Lambda, x) \subset \mathcal{O}_F\},$$

and $\langle\ ,\ \rangle$ induces a perfect $\mathbb{Z}_p$-bilinear pairing

$$\Lambda \times \widehat{\Lambda} \longrightarrow \mathbb{Z}_p.$$

An $\mathcal{O}_F$-lattice chain $\mathcal{L}$ is called *self-dual* if $\Lambda \in \mathcal{L}$ implies $\widehat{\Lambda} \in \mathcal{L}$. The standard lattice chain defined above is self-dual with $\widehat{\Lambda}_i = \Lambda_{-i}$.

Now we consider the group $G := GU(\phi)$ of unitary similitudes for the quadratic extension $F = K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ of $\mathbb{Q}_p$ as in [11, pp.25]: Over F we have the standard splitting

$$G_F \xrightarrow[\sim]{(f,c)} GL_n \times \mathbb{G}_m \tag{2.1}$$

where $c : G_F \to \mathbb{G}_m$ is the similitude character and $f : G_F \to GL_n$ is given on R-points by the map on matrix entries

$$\begin{aligned} R \otimes_{\mathbb{Q}_p} F &\longrightarrow R \\ x \otimes y &\longmapsto xy \end{aligned}$$

for an F-algebra R. Given a partition $n = r + s$, the pair (r, s) is referred to as the *signature*. We write $(1^{(r)}, 0^{(s)})$ for the cocharacter

$$x \longmapsto \mathrm{diag}(\underbrace{x, \ldots, x}_{r}, \underbrace{1, \ldots, 1}_{s})$$

of the standard maximal torus D of diagonal matrices in GL_n. Via (2.1), we can regard the cocharacter $\mu := (1^{(s)}, 0^{(r)}; 1)$ of $D \times \mathbb{G}_m$ as a geometric cocharacter of G. We denote by $\{\mu\}$ its geometric conjugacy class and by E the reflex field of $\{\mu\}$. Then $E = \mathbb{Q}_p$ if $r = s$ and $E = F$ otherwise.

Let $m = \lfloor n/2 \rfloor$. Let I be a nonempty subset of $\{1, \ldots, m\}$ such that

$$\text{if } n \text{ is even and } m - 1 \in I \text{ then } m \in I.$$

We define the functor M_I^{naive} on the category of $\mathcal{O}_E$-schemes as in [10]: A point of M_I^{naive} with values in an $\mathcal{O}_E$-scheme S is given by an $\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-submodule

$$\mathcal{F}_j \subset \Lambda_j \otimes_{\mathbb{Z}_p} \mathcal{O}_S$$

for each $j \in \{\pm i + tn \mid i \in I \text{ and } t \in \mathbb{Z}\}$. The following conditions (a)-(d) are imposed.

(a) As an $\mathcal{O}_S$-module, $\mathcal{F}_j$ is locally on S a direct summand of rank n.

(b) For each $j < j'$, there is a commutative diagram

$$\begin{array}{ccc} \Lambda_j \otimes_{\mathbb{Z}_p} \mathcal{O}_S & \longrightarrow & \Lambda_{j'} \otimes_{\mathbb{Z}_p} \mathcal{O}_S \\ \cup & & \cup \\ \mathcal{F}_j & \longrightarrow & \mathcal{F}_{j'} \end{array}$$

where the top horizontal map is induced by the inclusion $\Lambda_j \subset \Lambda_{j'}$, and for each j, the isomorphism $\pi : \Lambda_j \to \Lambda_{j-n}$ induces an isomorphism of $\mathcal{F}_j$ with $\mathcal{F}_{j-n}$.

(c) We have $\mathcal{F}_{-j} = \mathcal{F}_j^{\perp}$ where $\mathcal{F}_j^{\perp}$ is the orthogonal complement of $\mathcal{F}_j$ under the natural perfect pairing

$$(\Lambda_{-j} \otimes_{\mathbb{Z}_p} \mathcal{O}_S) \times (\Lambda_j \otimes_{\mathbb{Z}_p} \mathcal{O}_S) \to \mathcal{O}_S.$$

(d) Here we consider the action of $\pi \otimes 1 \in \mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ as an $\mathcal{O}_S$-linear operator on $\mathcal{F}_j$. The characteristic polynomial equals

$$\det((T \cdot \mathrm{id} - \pi \otimes 1) \mid \mathcal{F}_j) = (T - \pi)^s \cdot (T + \pi)^r \in \mathcal{O}_E[T].$$

M_I^{naive} is representable by a projective scheme and called the *naive local model* associated to the group G, the signature (r, s), and the lattice chain $\{\Lambda_j \mid j = \pm i + tn$ for $i \in I$ and $t \in \mathbb{Z}\}$. Note that this lattice chain is periodic and self-dual. In general naive local models fail to be flat, as shown in [9]. To force flatness, we take the scheme-theoretic flat closure of the generic fiber in M_I^{naive} and define it as the *honest local model* M_I^{loc}.

Remark 2.1. ([11, Rmk 2.28], [10, §1.2.3]) Recall that $m = \lfloor n/2 \rfloor$. Up to $G(\mathbb{Q}_p)$-conjugacy, the following

- n is odd and $I = \{0\}$ or $I = \{m\}$,
- n is even and $I = \{m\}$,

are all the cases that correspond to special maximal parahoric level structure, i.e. the parahoric subgroup defined as the stabilizer of the lattice chain $\{\Lambda_j \mid j = \pm i + tn$ for $i \in I$ and $t \in \mathbb{Z}\}$

in $G(\mathbb{Q}_p)$ corresponds to a vertex in the building which is special in the sense of Bruhat-Tits theory.

Theorem 2.2. ([10, Thm 5.1]) Let $I = \{0\}$ if n is odd and $I = \{n/2\}$ if n is even. For any signature (r, s), the special fiber of M_I^{loc} is irreducible, reduced, normal, and Frobenius split with only rational singularities.

One way of obtaining a moduli-theoretic description of M_I^{loc} is by strengthening the naive formulation of the moduli problem. The condition (d) fails to force a condition on the reduced special fiber. Motivated by this, the following additional condition was introduced to the moduli problem in [9]:

(e) (*wedge condition*) If $r \neq s$, we have

$$\wedge^{r+1}(\pi \otimes 1 - 1 \otimes \pi \mid \mathcal{F}_j) = 0 \quad \text{and} \quad \wedge^{s+1}(\pi \otimes 1 + 1 \otimes \pi \mid \mathcal{F}_j) = 0.$$

The *wedge local model* $M_I^{\wedge}$ is the closed subscheme of M_I^{naive} defined by the conditions (a) - (e). The wedge and naive local models have the same generic fiber.

For the rest of this section, we focus on the case n is even and $I = \{n/2\}$. When $r \neq 0$ or $s \neq 0$, it is known that the scheme $M_{\{n/2\}}^{\wedge}$ is not topologically flat, i.e. the generic fiber in $M_{\{n/2\}}^{\wedge}$ is not dense [11, Rmk 2.32]. In [10] the *spin condition* was proposed in addition to the wedge condition, and we define M_I^{spin} as the closed subscheme of $M_I^{\wedge}$ by imposing the spin condition. We remark that the spin condition is defined in general for any n and I, but its formulation is quite complicated so that we only describe its result in our case n is even and $I = \{n/2\}$; see [10, §7.2] for the general formulation of the spin condition. The following is taken from [11, pp.30]: When n is even and $I = \{n/2\}$, the perfect pairing

$$\Lambda_{n/2} \times \Lambda_{n/2} \xrightarrow{id \times \pi} \Lambda_{n/2} \times \Lambda_{-n/2} \xrightarrow{\langle\, , \rangle} \mathcal{O}_{\mathbb{Q}_p}$$

is split symmetic. Hence $M_{\{n/2\}}^{naive}$ naturally embeds as a closed subscheme of $OGr(n, 2n)_{\mathcal{O}_E}$. Then the spin condition amounts exactly to intersecting $M_{\{n/2\}}^{naive}$ with the connected component of $OGr(n, 2n)_{\mathcal{O}_E}$ marked by the common generic fiber of $M_{\{n/2\}}^{naive}$ and $M_{\{n/2\}}^{\wedge}$. It is shown in [10] that $M_{\{n/2\}}^{spin}$ and $M_{\{n/2\}}^{naive}$ have the same generic fiber and that $M_{\{n/2\}}^{spin}$ is topologically flat.

There is a question whether the wedge condition and the spin condition cut out the flat closure in $M^{naive}_{\{n/2\}}$. Our goal is to show that $M^{spin}_{\{n/2\}}$ is flat over Spec($\mathcal{O}_E$), i.e. $M^{loc}_{\{n/2\}} = M^{spin}_{\{n/2\}}$. Then it is related to a conjecture about schemes of matrices given by explicit matrix identities:

Conjecture 2.3. ([10, Conj. 5.2]) Let $n = r + s$ be a partition such that both r and s are even. Consider the scheme of matrices X in $M_{n\times n}$ over Spec k described by

$$X^2 = 0, \quad X^T = -JXJ, \quad \mathrm{char}_X(T) = T^n$$

and

$$\wedge^{s+1}X = 0, \quad \wedge^{r+1}X = 0 \quad \text{when } r \neq s$$

Then this scheme is reduced.

Here J is the matrix given at (1.2). In general, it is known that if a scheme over a discrete valuation ring is topologically flat and its special fiber is reduced, then the scheme is flat. In [10], Pappas and Rapoport show that the scheme X appearing in Conjecture 2.3 is an open affine chart in the special fiber of $M^{spin}_{\{n/2\}}$, which meets every orbit in $M^{spin}_{\{n/2\}}$ under a natural group action on $M^{spin}_{\{n/2\}}$. Since $M^{spin}_{\{n/2\}}$ is already proven to be topologically flat, we conclude that Conjecture 2.3 implies the flatness of $M^{spin}_{\{n/2\}}$.

3 The Relations

Now we reset all our notations except for the scheme V defined in the introduction. Let $n = 2r$ be an even integer with $n \geq 4$ and F be an arbitrary field. We denote by $\mathbf{n}$ the set $\{1, 2, \ldots, n\}$ and, for $1 \leq i \leq r$, we often use the symbol $\bar{i}$ in place of $r + i$. That is,

$$\mathbf{n} = \{1, 2, \ldots, r, \overline{1}, \overline{2}, \ldots, \overline{r}\}.$$

Let C be any F-algebra. Let E be a free C-module of rank n with a basis $\{e_1, \ldots, e_r, e_{\overline{1}}, \ldots, e_{\overline{r}}\}$. We endow E with a nondegenerate antisymmetric bilinear form $\langle\ ,\ \rangle$ such that

$$\langle e_i, e_{\bar{j}}\rangle = \delta_{ij} = -\langle e_{\bar{j}}, e_i\rangle, \quad \langle e_i, e_j\rangle = 0, \quad \langle e_{\bar{i}}, e_{\bar{j}}\rangle = 0$$

for any $i, j \in \{1, \ldots, r\}$. In other words, $\langle\ ,\ \rangle$ is represented by the matrix J, (1.2), with respect to the basis $\{e_1, \ldots, e_r, e_{\overline{1}}, \ldots, e_{\overline{r}}\}$.

For each $1 \leq k \leq r$ and $t \leq \lfloor k/2 \rfloor$, we introduce a homomorphism

$$\Phi_{k,t} : \bigwedge^{k} E \to \bigwedge^{k-2t} E$$

defined by

$$\Phi_{k,t}(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \sum_{\sigma} sgn(\sigma) \langle v_{\sigma(1)}, v_{\sigma(2)} \rangle \cdots \langle v_{\sigma(2t-1)}, v_{\sigma(2t)} \rangle \, v_{\sigma(2t+1)} \wedge \cdots \wedge v_{\sigma(k)},$$

where σ runs through all permutations of $\{1, 2, \ldots, k\}$ such that

$$\sigma(1) < \sigma(2), \quad \ldots, \quad \sigma(2t-1) < \sigma(2t), \quad \text{and} \quad \sigma(2t+1) < \sigma(2t+2) < \cdots < \sigma(k).$$

It is easy to see that $\Phi_{k,t}$ is well-defined, and in fact, $\Phi_{k,t}$ can be obtained recursively:

$$\Phi_{k,t} = \Phi_{k-2t+2,1} \circ \cdots \circ \Phi_{k-2,1} \circ \Phi_{k,1}. \tag{3.1}$$

Assume that the scheme V is defined over F. $\Phi_{k,t}$ is the main tool to find relations between pfaffians in the coordinate ring of V, cf.[3, p.5]. One goal of this section is to achieve the relation (3.5) below, which is the analog of the equation given in [3, Prop.1.8] if we substitute pfaffians for minors (with fixed column indices). Some of the proofs and definitions in this section closely follow [3, pp.5-8].

Lemma 3.1. For any $n \times n$ matrix Y in $V(C)$, we have

$$2 \sum_{h=1}^{r} Y_{h\overline{h}} = 0.$$

Proof. $-JY$ has trace 0 since $\text{char}_{(-JY)}(T) = T^n$. Then

$$0 = \text{tr}(-JY) = -\sum_{h=1}^{r} Y_{\overline{h}h} + \sum_{h=1}^{r} Y_{h\overline{h}} = -\sum_{h=1}^{r}(-Y_{h\overline{h}}) + \sum_{h=1}^{r} Y_{h\overline{h}} = 2\sum_{h=1}^{r} Y_{h\overline{h}} . \qquad \square$$

Lemma 3.2. Assume $k = 2m$ is an even number and $Y \in V(C)$. Let

$$w := \sum_{i,j\in\mathbf{n}} Y_{ij}\, e_i \wedge e_j.$$

Then $\Phi_{k,t}(w^m) = 0$ for any $t \le m$. Here the m-th power is taken in the exterior algebra $\bigwedge E$.

Proof. It suffices to show $\Phi_{k,1}(w^m) = 0$ by (3.1).

Case $m = 1$: By Lemma 3.1,

$$\begin{aligned}
\Phi_{2,1}(w) &= \Phi_{2,1}(\sum_{i,j\in\mathbf{n}} Y_{ij}\, e_i \wedge e_j) \\
&= \sum_{i,j\in\mathbf{n}} Y_{ij}\langle e_i, e_j\rangle \\
&= \sum_{h=1}^{r} Y_{h\overline{h}}\langle e_h, e_{\overline{h}}\rangle + \sum_{h=1}^{r} Y_{\overline{h}h}\langle e_{\overline{h}}, e_h\rangle \\
&= 2\sum_{h=1}^{r} Y_{h\overline{h}} \\
&= 0.
\end{aligned}$$

Case $m = 2$:

$$\begin{aligned}
\Phi_{4,1}(w^2) &= \Phi_{4,1}\left((\sum_{i,j\in\mathbf{n}} Y_{ij} e_i \wedge e_j)(\sum_{i',j'\in\mathbf{n}} Y_{i'j'} e_{i'} \wedge e_{j'}) \right) \\
&= \sum_{i,j,i',j'\in\mathbf{n}} Y_{ij} Y_{i'j'} \Phi_{4,1}(e_i \wedge e_j \wedge e_{i'} \wedge e_{j'}).
\end{aligned} \tag{3.2}$$

By definition,

$$\begin{aligned}
\Phi_{4,1}(e_i \wedge e_j \wedge e_{i'} \wedge e_{j'}) = \langle e_i, e_j\rangle e_{i'} \wedge e_{j'} + \langle e_{i'}, e_{j'}\rangle e_i \wedge e_j - \langle e_i, e_{i'}\rangle e_j \wedge e_{j'} - \langle e_j, e_{j'}\rangle e_i \wedge e_{i'} \\
+ \langle e_i, e_{j'}\rangle e_j \wedge e_{i'} + \langle e_j, e_{i'}\rangle e_i \wedge e_{j'}.
\end{aligned}$$

Then we compute the sum (3.2) with each of these six terms. It turns out that each sum is equal to zero, so we get $\Phi_{4,1}(w^2) = 0$. We use Lemma 3.1 for the first one and the second one.

The first sum is

$$
\begin{aligned}
\sum_{i,j,i',j'} Y_{ij} Y_{i'j'} \langle e_i, e_j \rangle e_{i'} \wedge e_{j'} &= \sum_{i',j'} \left(\sum_{i,j} Y_{ij} \langle e_i, e_j \rangle \right) Y_{i'j'} \, e_{i'} \wedge e_{j'} \\
&= \sum_{i',j'} \left(2 \sum_{h=1}^{r} Y_{h\bar{h}} \right) Y_{i'j'} \, e_{i'} \wedge e_{j'} \\
&= 0,
\end{aligned}
$$

and the second sum can similary be shown to be zero.

For the third one, we use the condition $Y^T J Y = 0$:

$$
\begin{aligned}
\sum_{i,j,i',j'} Y_{ij} Y_{i'j'} \langle e_i, e_{i'} \rangle e_j \wedge e_{j'} &= \sum_{j,j'} \left(\sum_{i,i'} Y_{ij} \langle e_i, e_{i'} \rangle Y_{i'j'} \right) e_j \wedge e_{j'} \\
&= \sum_{j,j'} \left(\sum_{i,i'} (Y^T)_{ji} \langle e_i, e_{i'} \rangle Y_{i'j'} \right) e_j \wedge e_{j'} \\
&= \sum_{j,j'} \ (Y^T J Y)_{jj'} \ \ e_j \wedge e_{j'} \\
&= 0.
\end{aligned}
$$

Recall that Y is a skew-symmetric matrix, i.e. $Y = -Y^T$. Hence we have $0 = Y^T J Y = Y J Y^T = Y^T J Y^T = Y J Y$, and we will use this equation to show that the fourth, the fifth, and the sixth sums are again equal to zero. The computations are analogous to the third one and listed below:

$$
\begin{aligned}
\sum_{i,j,i',j'} Y_{ij} Y_{i'j'} \langle e_j, e_{j'} \rangle e_i \wedge e_{i'} &= \sum_{i,i'} \left(\sum_{j,j'} Y_{ij} \langle e_j, e_{j'} \rangle Y_{i'j'} \right) e_i \wedge e_{i'} \\
&= \sum_{i,i'} \left(\sum_{j,j'} Y_{ij} \langle e_j, e_{j'} \rangle (Y^T)_{j'i'} \right) e_i \wedge e_{i'} \\
&= \sum_{i,i'} \ (Y J Y^T)_{ii'} \, e_i \wedge e_{i'} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i,j,i',j'} Y_{ij}Y_{i'j'}\langle e_i, e_{j'}\rangle e_j \wedge e_{i'} &= \sum_{j,i'} \Big(\sum_{i,j'} Y_{ij}\langle e_i, e_{j'}\rangle Y_{i'j'} \Big) e_j \wedge e_{i'} \\
&= \sum_{j,i'} \Big(\sum_{i,j'} (Y^T)_{ji}\langle e_i, e_{j'}\rangle (Y^T)_{j'i'} \Big) e_j \wedge e_{i'} \\
&= \sum_{j,i'} \ (Y^T J Y^T)_{ji'} \, e_j \wedge e_{i'} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i,j,i',j'} Y_{ij}Y_{i'j'}\langle e_j, e_{i'}\rangle e_i \wedge e_{j'} &= \sum_{i,j'} \Big(\sum_{j,i'} Y_{ij}\langle e_j, e_{i'}\rangle Y_{i'j'} \Big) e_i \wedge e_{j'} \\
&= \sum_{i,j'} \ (YJY)_{ij'} \, e_i \wedge e_{j'} \\
&= 0.
\end{aligned}
$$

Case $m \geq 2$: The general case can be achieved from the first two cases as

$$
\begin{aligned}
\Phi_{k,1}(w^m) &= \binom{m}{2} \Big\{ \Phi_{4,1}(w^2) \wedge w^{m-2} - 2 \cdot \Phi_{2,1}(w) \cdot w^{m-1} \Big\} + m \cdot \Phi_{2,1}(w) \cdot w^{m-1} \\
&= \binom{m}{2} \Phi_{4,1}(w^2) \wedge w^{m-2} + (-m^2 + 2m) \cdot \Phi_{2,1}(w) \cdot w^{m-1} \\
&= 0.
\end{aligned}
$$

$\square$

Notation 3.3. When $I = \{i_1, i_2, \ldots, i_l\}$ is a subset of $\mathbf{n}$ with $i_1 < i_2 < \cdots < i_l$, let e^I denote the vector

$$
e^I := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l} \, .
$$

We define a basis for $\bigwedge^k E$ as in [3, p.5]. Let $P = \{p_1, \ldots, p_s\}$ and $Q = \{q_1, \ldots, q_{k-s}\}$ be two subsets of $\{1, \ldots, r\}$ with $|P| + |Q| = k$. Let $\overline{Q}$ denote the set $\{\overline{q_1}, \ldots, \overline{q_{k-s}}\}$. A vector $e^{P,\overline{Q}} \in \bigwedge^k E$ is defined as follows:

(A) If $P \cap Q = \emptyset$, then $e^{P,\overline{Q}} := e^P \wedge e^{\overline{Q}}$.

(B) If $P \cap Q = \Gamma$ where $\Gamma = \{\gamma_1 < \cdots < \gamma_\lambda\}$, then $e^{P,\overline{Q}} := e_{\gamma_1} \wedge e_{\overline{\gamma_1}} \wedge \cdots \wedge e_{\gamma_\lambda} \wedge e_{\overline{\gamma_\lambda}} \wedge e^{P \smallsetminus \Gamma} \wedge e^{\overline{Q \smallsetminus \Gamma}}$.

Clearly $\{e^{P,\overline{Q}} \mid P, Q \subset \{1, \ldots, r\},\ |P| + |Q| = k\}$ forms a basis of $\bigwedge^k E$.

Example 3.4. Let $P = \{1, 2, 5\}$ and $Q = \{2, 3, 4\}$. Then

$$e^{P,\overline{Q}} = e^{\{1,2,5\},\{\overline{2},\overline{3},\overline{4}\}} = e_2 \wedge e_{\overline{2}} \wedge e_1 \wedge e_5 \wedge e_{\overline{3}} \wedge e_{\overline{4}}.$$

Lemma 3.5. (1) If $t > |P \cap Q|$ then

$$\Phi_{k,t}(e^{P,\overline{Q}}) = 0.$$

(2) If $t \leq |P \cap Q|$ then

$$\Phi_{k,t}(e^{P,\overline{Q}}) = t! \sum_{\Gamma_t} e^{P \smallsetminus \Gamma_t,\, \overline{Q \smallsetminus \Gamma_t}}$$

where Γ_t runs through all size t subsets of $\Gamma = P \cap Q$.

Proof. Both assertions are easily seen from the definitions. $\square$

As in [3, p.6], we follow the convention of putting $e^{P \smallsetminus \Gamma_t,\, \overline{Q \smallsetminus \Gamma_t}} = 0$ when $\Gamma_t \not\subset P \cap Q$. Hence, we can write

$$\Phi_{k,t}(e^{P,\overline{Q}}) = t! \sum_{|\Gamma_t| = t} e^{P \smallsetminus \Gamma_t,\, \overline{Q \smallsetminus \Gamma_t}}.$$

Let $A = (A_{ij})$ be a $2s \times 2s$ skew-symmetric matrix. The pfaffian of A is a polynomial $Pf(A)$ in the entries of A defined by

$$Pf(A) = \sum_{\sigma} sgn(\sigma) \prod_{i=1}^{s} A_{\sigma(2i-1)\sigma(2i)},$$

where the summation is over all permutations σ of $\{1, 2, \ldots, 2s\}$ such that

$$\sigma(1) < \sigma(3) < \cdots < \sigma(2s-1) \quad \text{and} \quad \sigma(2i-1) < \sigma(2i) \text{ for } i = 1, 2, \ldots, s.$$

It is well known that $Pf(A)^2 = \det(A)$. If A has entries in C and $\{e_1, e_2, \ldots, e_{2s}\}$ is the standard basis of C^{2s}, then $Pf(A)$ satisfies

$$\big(\sum_{i<j} A_{ij}\, e_i \wedge e_j\big)^s = s!\, Pf(A)\, e_1 \wedge e_2 \wedge \cdots \wedge e_{2s};$$

see [2, §5.2].

For a given $Y \in V(C)$, let $[i_1, i_2, \ldots, i_{2l}]$ denote the pfaffian of the principal submatrix of Y indexed by $i_1, i_2, \ldots, i_{2l} \in \mathbf{n}$. Then for any $m \leq r$, we have

$$(\sum_{i,j\in\mathbf{n}} Y_{ij} e_i \wedge e_j)^m = 2^m (\sum_{\substack{i,j\in\mathbf{n}\\ i<j}} Y_{ij} e_i \wedge e_j)^m = 2^m m! \sum_{i_1<i_2<\cdots<i_{2m}} [i_1, i_2, \ldots, i_{2m}] e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{2m}}$$

where the rightmost sum is among all size $2m$ subsets of $\mathbf{n}$. Recall that $e^{P,\overline{Q}}$ is a wedge product of e_i, $i \in P \cup \overline{Q}$, but the indices are not necessarily in increasing order (see Notation 3.3 (B)). It is convenient to define $[P, \overline{Q}]$ likewise so that we have the identity

$$(\sum_{i,j\in\mathbf{n}} Y_{ij}\, e_i \wedge e_j)^m = 2^m m! \sum_{\substack{P,Q\subset\{1,\cdots,r\}\\ |P|+|Q|=2m}} [P, \overline{Q}]\, e^{P,\overline{Q}}\,. \tag{3.3}$$

That is, $[P, \overline{Q}]$ denotes the pfaffian of the $2m \times 2m$ principal submatrix of Y indexed by $P \cup \overline{Q}$, but the order of indices coincides with that of $e^{P,\overline{Q}}$.

Example 3.6. Let $P = \{1, 2, 5\}$ and $Q = \{2, 3, 4\}$. Then

$$[P, \overline{Q}] = [\{1, 2, 5\}, \{\overline{2}, \overline{3}, \overline{4}\}] = [2, \overline{2}, 1, 5, \overline{3}, \overline{4}]$$

($= -[1, 2, 5, \overline{2}, \overline{3}, \overline{4}]$ since $(2, \overline{2}, 1, 5, \overline{3}, \overline{4})$ is an odd permutation of $(1, 2, 5, \overline{2}, \overline{3}, \overline{4})$).

Lemma 3.7. Assume $k = 2m$ is an even integer, and $1 \leq t \leq m \leq r$. Let P' and Q' be two fixed subsets of $\mathbf{r} := \{1, 2, \ldots, r\}$ with $|P'| + |Q'| = k - 2t$. Then for any $Y \in V(C)$,

$$2^m m! t! \sum_{\Gamma_t} [P' \cup \Gamma_t, \overline{Q' \cup \Gamma_t}] = 0$$

where Γ_t runs through all size t subsets of $\mathbf{r} \setminus (P' \cup Q')$.

In particular, if F is a field with char $F=0$ or char $F>r$ then

$$\sum_{\substack{\Gamma_t \subset \mathbf{r} \setminus (P' \cup Q') \\ |\Gamma_t| = t}} [P' \cup \Gamma_t, \overline{Q' \cup \Gamma_t}] = 0. \tag{3.4}$$

Proof. This lemma is based on [3, Prop.1.7] and follows a similar proof structure. By Lemma 3.2 and equation (3.3), we have

$$0 = \Phi_{k,t}\Big((\sum_{i,j \in \mathbf{n}} Y_{ij} e_i \wedge e_j)^m \Big) = \Phi_{k,t}\Big(2^m m! \sum_{\substack{P,Q \subset \mathbf{r} \\ |P|+|Q|=k}} [P, \overline{Q}]\, e^{P,\overline{Q}} \Big).$$

Then

$$\begin{aligned}
\Phi_{k,t}\Big(2^m m! \sum_{\substack{P,Q \subset \mathbf{r} \\ |P|+|Q|=k}} [P, \overline{Q}]\, e^{P,\overline{Q}} \Big) &= 2^m m! \sum_{\substack{P,Q \subset \mathbf{r} \\ |P|+|Q|=k}} [P, \overline{Q}]\, \Phi_{k,t}(e^{P,\overline{Q}}) \\
&= 2^m m! \sum_{\substack{P,Q \subset \mathbf{r} \\ |P|+|Q|=k}} [P, \overline{Q}] \Big(t! \sum_{|\Gamma_t|=t} e^{P \setminus \Gamma_t,\, \overline{Q \setminus \Gamma_t}} \Big) \\
&= 2^m m! t! \sum_{\substack{P',Q' \subset \mathbf{r} \\ |P'|+|Q'|=k-2t}} \Big(\sum_{\substack{\Gamma_t \subset \mathbf{r} \setminus (P' \cup Q') \\ |\Gamma_t| = t}} [P' \cup \Gamma_t, \overline{Q' \cup \Gamma_t}] \Big) e^{P',\overline{Q'}}
\end{aligned}$$

by putting $P' = P \setminus \Gamma_t$, $Q' = Q \setminus \Gamma_t$ and changing the order of summation. Since $e^{P',\overline{Q'}}$ are linearly independant, this proves the lemma. When char $F=0$ or char $F>r$, $2^m m! t! \in C^\times$ so we get (3.4). □

For the rest of this section, we assume that F is a field with char $F=0$ or char $F>r$.

Proposition 3.8. Let $[P' \cup \Gamma, \overline{Q' \cup \Gamma}]$ be a fixed pfaffian of $Y \in V(C)$ with $\Gamma \subset \{1, 2, \ldots, r\} \setminus (P' \cup Q')$ and $\Gamma \neq \varnothing$. (P' and Q' are not necessarily disjoint.) Then

$$[P' \cup \Gamma, \overline{Q' \cup \Gamma}] - (-1)^{|\Gamma|} \sum_{\Gamma'} [P' \cup \Gamma', \overline{Q' \cup \Gamma'}] = 0 \tag{3.5}$$

where Γ' runs over the subsets of $\{1, 2, \ldots, r\} \setminus (P' \cup Q' \cup \Gamma)$ with $|\Gamma'| = |\Gamma|$.

Proof. The proof is essentially the same as that of [3, Prop.1.8] if we substitute pfaffians for the minors (with fixed column indices $h_1, \ldots, h_k$) and apply (3.4) as a replacement for [3, (1.7)]. $\square$

Corollary 3.9. For $Y \in V(C)$, any pfaffian $[P, \overline{Q}]$ with $|P| + |Q| > r$ vanishes.

Proof. Let $[P, \overline{Q}]$ be such a pfaffian. Then clearly $P \cap Q \neq \varnothing$. We define

$$\Gamma = P \cap Q, \quad P' = P \setminus \Gamma, \quad Q' = Q \setminus \Gamma,$$

so that $[P, \overline{Q}] = [P' \cup \Gamma, \overline{Q' \cup \Gamma}]$. Since $|P'| + |Q'| + 2|\Gamma| > r$, there is no $\Gamma' \subset \{1, 2, \ldots, r\} \setminus (P' \cup Q' \cup \Gamma)$ such that $|\Gamma'| = |\Gamma|$. By Proposition 3.8, we get

$$[P' \cup \Gamma, \overline{Q' \cup \Gamma}] = 0. \qquad \square$$

4 Symplectic standard even-tableaux

We define a tableau as in [5, p.117]. Let N be a positive integer and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of N, i.e.

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = N.$$

The *diagram* $D(\lambda)$ of λ is defined as the set $\{(s,t) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq s \leq k,\ 1 \leq t \leq \lambda_s\}$. Recall that $\mathbf{n}$ denotes the set $\{1, 2, \ldots, r, \overline{1}, \overline{2}, \ldots, \overline{r}\}$. A *tableau* of shape λ with entries in $\mathbf{n}$ is a map $T : D(\lambda) \to \mathbf{n}$, depicted by its array of values

$$\begin{array}{lllll} T(1,1) & T(1,2) & \cdots & \cdots & T(1,\lambda_1) \\ T(2,1) & T(2,2) & \cdots & T(2,\lambda_2) & \\ T(3,1) & T(3,2) & \cdots & & \\ \vdots & \vdots & & & \end{array}$$

In this document, tableaux have entries in $\mathbf{n}$ unless specified otherwise. We order the indices of $\mathbf{n}$ by

$$\overline{1} \prec 1 \prec \overline{2} \prec 2 \prec \cdots \prec \overline{r} \prec r. \tag{4.1}$$

(This is different from the natural numerical order.) A tableau T is called *standard* if its entries strictly increase along each row and weakly increase down each column with respect to the order (4.1). Now we give the definition of *symplectic standard* tableaux following King [8]; note however that the role of rows and columns are exchanged in definitions of standard tableaux and symplectic standard tableaux in [5] and [8]. (De Concini also defines symplectic standard tableaux in [3], but that definition is different from the one given here.)

Definition 4.1. A tableau T is called *symplectic standard* if

- it is standard, and
- the indices p and $\overline{p}$ appear only in the first p columns for $1 \le p \le r$.

Each row of a symplectic standard tableau must have length $\le r$ by definition.

Definition 4.2. If a tableau has no odd-sized rows, we call it an *even-tableau* (and *even-tableaux* for the plural form). In other words, if T is an even-tableau of shape $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, then all λ_i are even.

Let R be a commutative ring with unity and let $Y = (Y_{ij})$ be an $n \times n$ skew-symmetric matrix of indeterminates, i.e.

- $Y_{ij} = -Y_{ji}$ if $i < j$, and
- $Y_{ii} = 0$ for all i.

Let $R[Y_{ij}]_{i<j}$ denote the polynomial ring $R[Y_{ij} : 1 \le i < j \le n]$. When $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a partition of an even integer $2m$ such that all λ_i are even, we associate a tableau T of shape λ with the product of pfaffians evaluated at the generic skew-symmetric matrix Y, denoted by $[T]$,

$$[T] := \prod_{i=1}^{k} [T(i,1), T(i,2), \dots, T(i,\lambda_i)].$$

Clearly $[T]$ can be considered as a degree m homogeneous polynomial in $R[Y_{ij}]_{i<j}$. The combinatorial structure of $R[Y_{ij}]_{i<j}$ is observed by the following facts; see [4, Ch.6] for the proof.

Theorem 4.3. (De Concini and Procesi, 1976)

(1) The products of pfaffians indexed by the standard even-tableaux form an R-basis of $R[Y_{ij}]_{i<j}$.

(2) For any given even-tableau T, the standard representation of $[T]$ can be achieved by iterated use of the following relation

$$\begin{aligned}[a_1,\dots,a_{2t}][b_1,\dots,b_{2s}] - \sum_{i=1}^{2t}[a_1,\dots,a_{i-1},b_1,a_{i+1},\dots,a_{2t}][a_i,b_2,\dots,b_{2s}]\\ = \sum_{j=2}^{2s}(-1)^{j-1}[b_2,\dots,b_{j-1},b_{j+1},\dots,b_{2s}][b_j,b_1,a_1,\dots,a_{2t}].\end{aligned}$$

Remark 4.4. Consider the natural numerical order given on the indices of $\mathbf{n}$:

$$1 < 2 < \cdots < r < \overline{1} < \overline{2} < \cdots < \overline{r}. \tag{4.2}$$

Recall that the definition of the standard tableaux depends on the order (4.1) given on $\mathbf{n}$. There is another R-basis of $R[Y_{ij}]_{i<j}$ indexed by '$<$-standard' even-tableaux. More precisely, the map $T \mapsto [T]$ from

$$\left\{ T \;\middle|\; \begin{array}{l} T \text{ is an even-tableau such that the entries of } T \text{ strictly increase along each row} \\ \text{and weakly increase down each column with respect to the order } (4.2) \end{array} \right\}$$

into $R[Y_{ij}]_{i<j}$ is injective and its image forms an R-basis of $R[Y_{ij}]_{i<j}$.

In [5], the *type* of a tableau T is defined as the $2r$-tuple of integers $(a_1, b_1, \dots, a_r, b_r)$ where

$$a_p := |T^{-1}(\overline{p})| = \text{the number of occurences of } \overline{p} \text{ in } T$$
$$b_p := |T^{-1}(p)| = \text{the number of occurences of } p \text{ in } T$$

for $1 \le p \le r$. We define a total order on the set of $2r$-tuples of integers by setting

$$(a_1, b_1, \dots, a_r, b_r) \trianglelefteq (a'_1, b'_1, \dots, a'_r, b'_r)$$

if $(b_r, a_r, \dots, b_1, a_1)$ is less than or equal to $(b'_r, a'_r, \dots, b'_1, a'_1)$ in the lexicographic order.

Remark 4.5. Note that the straightening relation in Theorem 4.3 (2) does not change the type of a given tableau. Hence for every even-tableau T, $[T]$ can be written as a linear sum

$[T] = c_1[T_1] + \cdots + c_k[T_k]$, $c_i \in R$, where all T_i are standard even-tableaux whose types are same as the type of T.

Since the coordinate ring $R[V]$ is a quotient ring of $R[Y_{ij}]_{i<j}$, we can naturally associate an even-tableau T with an element of $R[V]$, which is also denoted by $[T]$ by an abuse of notation. Note that $R[V]$ is a graded algebra over R. (The polynomials in Y_{ij}, $1 \leq i, j \leq n$, obtained from (1.1) are all homogeneous.) We denote the degree m homogeneous part of $R[V]$ by $R[V]_m$. Let $\mathcal{T}(m)$ be the set of all even-tableaux whose shapes are partitions of $2m$. Theorem 4.3 (1) implies the following:

Corollary 4.6. The set $\{[T] \in R[V] \mid T$ is a standard even-tableau in $\mathcal{T}(m)\}$ spans $R[V]_m$ over R.

The proof of next propostion closely follows that of the Symplectic Carter-Lusztig lemma from [5, p.119].

Proposition 4.7. Let F be a field with char $F = 0$ or char $F > r$. The set $\{[T] \mid T$ is a symplectic standard even-tableau$\}$ spans $F[V]$ over F.

Proof. Let $m \geq 1$ be chosen aribitrarily. It suffices to prove that $F[V]_m$ is spanned by

$$\{[T] \mid T \text{ is a symplectic standard even-tableau in } \mathcal{T}(m)\}. \tag{4.3}$$

Let S be the F-span of (4.3), and suppose that S is a proper subset of $F[V]_m$ for a contradiction. Since the set of all $[T]$, $T \in \mathcal{T}(m)$, spans $F[V]_m$, the set $\{T \in \mathcal{T}(m) \mid [T] \notin S\}$ is nonempty. We choose an even-tableau T in this set such that the type of T is as large as possible in the $\trianglelefteq$ ordering. By Remark 4.5, there must be a standard even-tableau T' such that $[T'] \notin S$ and the type of T' is same as the type of T. Replacing T by T', we may assume that T is standard.

Since $[T] \notin S$, T cannot be symplectic standard. Hence there is a position (l, h) such that $T(l, h) = u$ or $\overline{u}$ with $u < h$. Assume that h is minimal with this property. Say $T(l, h-1) = v$ or $\overline{v}$. Then

$$h - 1 \leq v \leq u < h.$$

Here $h - 1 \leq v$ is by the minimality of h, and $v \leq u$ is by the standardness of T. Hence we are forced to have $h-1=v=u$ and more precisely,

$$T(l,h-1)=\overline{h-1} \quad \text{and} \quad T(l,h)=h-1.$$

Let $\mathbf{r}$ denote the set $\{1,2,\ldots,r\}$ and define

$$\Gamma=\{p\in\mathbf{r}\mid \text{both } p \text{ and } \overline{p} \text{ occur among } T(l,1),T(l,2),\ldots,T(l,h)\},$$

$$A=\{p\in\mathbf{r}\mid \text{exactly one of } p \text{ or } \overline{p} \text{ occurs among } T(l,1),T(l,2),\ldots,T(l,h)\},$$

$$P'=\{p\in\mathbf{r}\mid p \text{ occurs in row } l \text{ of } T\}\smallsetminus\Gamma,$$

$$Q'=\{p\in\mathbf{r}\mid \overline{p} \text{ occurs in row } l \text{ of } T\}\smallsetminus\Gamma.$$

Then row l of T corresponds to $[P'\cup\Gamma,\overline{Q'\cup\Gamma}]$ up to sign, and $2|\Gamma|+|A|=h$. From Proposition 3.8, we have

$$[P'\cup\Gamma,\overline{Q'\cup\Gamma}]=(-1)^{|\Gamma|}\sum_{\Gamma'}[P'\cup\Gamma',\overline{Q'\cup\Gamma'}] \tag{4.4}$$

where Γ' runs over the subsets of $\mathbf{r}\smallsetminus(P'\cup Q'\cup\Gamma)$ such that $|\Gamma'|=|\Gamma|$. Any such Γ' does not have intersection with $A\cup\Gamma$ (a disjoint union) and $h=2|\Gamma|+|A|=|\Gamma'|+|\Gamma|+|A|$. Note that both A and Γ are subsets of $\{1,2,\ldots,h-1\}$. As a result, Γ' must contain an element greater than $h-1$. Let $T_{\Gamma'}$ denote the tableau obtained from T by replacing its row l with the one row tableau corresponding to $[P'\cup\Gamma',\overline{Q'\cup\Gamma'}]$. Then $[T]$ is equal to

$$(-1)^{|\Gamma|}\sum_{\substack{\Gamma'\subset\mathbf{r}-P'\cup Q'\cup\Gamma\\|\Gamma'|=|\Gamma|}}[T_{\Gamma'}]$$

up to sign by (4.4). Since the type of each $T_{\Gamma'}$ is strictly bigger than the type of T with respect to $\trianglelefteq$, all $[T_{\Gamma'}]$ are in S by the maximality of the type of T. Now $[T]$ is also in S, a contradiction. $\square$

Now we prove the linear independency of the set

$$\{[T]:T \text{ is a symplectic standard even-tableau}\}$$

over a field F of arbitrary characteristic. First, let us specify some matrices in $Sp(n,F)$. Let

$E_{i,j}$ denote the $n \times n$ matrix which has 1 at entry (i,j) and 0 at all the other entries. It is easy to check that the matrices

$$I_n - \mu\, E_{i,j} + \mu\, E_{\bar{j},\bar{i}}, \quad I_n + \mu\, E_{\bar{i},i}, \quad I_n + \mu\, E_{\bar{j},i} + \mu\, E_{\bar{i},j}$$

are in $Sp(n, F)$ if $\mu \in F$ and $i, j \in \{1, 2, \ldots, r\}$ with $i \neq j$.

Definition 4.8. We say that a given tableau is *canonical* if in each column j of the tableau, only $\bar{j}$ appears.

Here is an example of a canonical tableau:

$$\begin{array}{cccc} \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{1} & \bar{2} & & \\ \bar{1} & \bar{2} & & \end{array}$$

Clearly every canonical tableau is symplectic standard.

Proposition 4.9. Let F be a field of arbitrary characteristic.

(1) The map from the set of all canonical even-tableaux into $F[V]$, given by $T \mapsto [T]$, is injective and its image in $F[V]$ is linearly independent over F.

(2) The map from the set of all symplectic standard even-tableaux into $F[V]$, given by $T \mapsto [T]$, is injective and its image in $F[V]$ is linearly independent over F.

Proof. Without loss of generality, we may assume that F is infinite. First, we consider the case r is even. For any F-algebra C, $V(C)$ contains all $n \times n$ matrices of the form

$$\left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & A \end{array}\right]$$

where A is a $r \times r$ skew-symmetric matrix with entries in C. In other words, there is a closed embedding from the scheme of $r \times r$ skew-symmetric matrices into V. Let

$$\psi : F[V] \to F[X_{ij}]_{i<j} := F[X_{ij} : 1 \leq i < j \leq r]$$

be the corresponding ring homomorphism. Note that every pfaffian in $F[V]$ of the form $[\bar{1}, \bar{2}, \ldots, \bar{i}]$, $1 \leq i \leq r$, maps to the pfaffian $[1, 2, \ldots, i]$ (evaluated at the generic $r \times r$ skew-symmetric matrix $X = (X_{ij})$) in $F[X_{ij}]_{i<j}$. Let $\mathbf{r} := \{1, 2, \ldots, r\}$ and let $\eta : \mathbf{n} \to \mathbf{r}$ be a map sending $\bar{i}$ to i for $1 \leq i \leq r$. For any canonical Tableau T with entries in $\mathbf{n}$, $\eta \circ T$ is a tableau with entries in $\mathbf{r}$, and $[T] \in F[V]$ clearly maps to $[\eta \circ T] \in F[X_{ij}]_{i<j}$ under ψ. Moreover, if T_1 and T_2 are distinct canonical tableaux with entries in $\mathbf{n}$, then $\eta \circ T_1$ and $\eta \circ T_2$ are also distinct tableaux in the following set:

$$\left\{ T \;\middle|\; \begin{array}{l} T \text{ is an even-tableau with entries in } \mathbf{r} \text{ such that the entries strictly increase along each} \\ \text{row and weakly increase down each column with respect to the order } 1 \leq 2 \leq \cdots \leq r \end{array} \right\}$$

Then by Remark 4.4, the (composition) map from the set of all canonical even-tableaux into $F[X_{ij}]_{i<j}$, given by $T \mapsto \psi([T])$, is injective and its image in $F[X_{ij}]_{i<j}$ is linearly independent over F. This proves (1) when r is even.

Now suppose r is an odd integer. We consider the $n \times n$ matrices of the form

$$\left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & \begin{array}{c:c} B & \begin{array}{c} 0 \\ \vdots \end{array} \\ \hdashline 0 \ \cdots & 0 \end{array} \end{array}\right]$$

denoted by $M(B)$, where B is the submatrix consisting of the rows $\bar{1}, \bar{2}, \ldots, \overline{r-1}$ and columns $\bar{1}, \bar{2}, \ldots, \overline{r-1}$. When C is an F-algebra, $V(C)$ contains all $M(B)$ such that B is a $(r-1) \times (r-1)$ skew-symmetric matrix with entries in C. Thus, there is a closed embedding from the scheme of $(r-1) \times (r-1)$ skew-symmetric matrices into V. Since r is odd, $\bar{r}$ never appears as an entry of T if T is a canonical even-tableau with entries in $\mathbf{n}$. The rest of the proof is analogous to the case r is even.

The proof of (2) closely follows [12, pp.506-508]. Recall that $Sp(n, F)$ acts on V: if $g \in Sp(n, F)$ and $Y \in V(C)$, C is any F-algebra, (right) action is given by $Y \cdot g = g^T Y g$. There is an induced action of $Sp(n, F)$ on $F[V]$, which can be described as

$$(g \cdot f)(Y) = f(Y \cdot g) = f(g^T Y g)$$

for $g \in Sp(n, F)$ and $f \in F[V]$. For example, the action of $I_n + \mu\, E_{\bar{i},i}$ ($\mu \in F$) on the matrix Y is the transformation $i \to i + \mu\, \bar{i}$ applied to Y on both rows and columns. Then the result of the induced action of $I_n + \mu\, E_{\bar{i},i}$ on $[i_1, i_2, \ldots, i_l] \in F[V]$ depends on whether i or $\bar{i}$ appear among the indices $i_1, i_2, \ldots, i_l$. More precisely,

$$\begin{aligned}
&(I_n + \mu\, E_{\bar{i},i}) \cdot [i_1, i_2, \ldots, i_l] \\
&= \begin{cases} [i_1, i_2, \ldots, i_l] + \mu\, [i_1, i_2, \ldots, i_{k-1}, \bar{i}, i_{k+1}, \ldots, i_l] & \text{if } i = i_k \text{ and } \bar{i} \text{ does not appear,} \\ [i_1, i_2, \ldots, i_l] & \text{if } i \text{ does not appear,} \\ [i_1, i_2, \ldots, i_l] & \text{if both } i \text{ and } \bar{i} \text{ appear.} \end{cases}
\end{aligned}$$

Finally, if $I_n + \mu\, E_{\bar{i},i}$ acts on $[T] \in F[V]$ for some even-tableau T, then we get a polynomial in μ (with coefficients in $F[V]$) of degree equal to the number of the rows of T in which i appears and $\bar{i}$ does not appear. The leading coefficient of this polynomial is $[T']$, where T' is a tableau obtained from T by replacing i with $\bar{i}$ in every row of T containing i but not $\bar{i}$.

Now assume that there is a dependence relation over F,

$$\sum_{i=1}^{s} c_i [T_i] = 0, \tag{4.5}$$

where the T_i are distinct symplectic standard even-tableaux. Some T_i is not canonical by (1), and therefore $T_i(u, v) \neq \bar{v}$ must hold for some i and some u, v. Let p be the entry $T_i(u, v)$ as small as possible with this property (with respect to the order (4.1)), i.e.

$$p := \min\{T_i(u,v) \mid \exists \text{ a tableau } T_i \text{ in } (4.5) \text{ and a position } (u,v) \text{ such that } T_i(u,v) \neq \overline{v}\}.$$

Then we define j and h in the following way:

$$\begin{aligned} j &:= \text{the minimum of the column indices where } p \text{ appear.} \\ &= \min\{v \mid \exists \text{ a tableau } T_i \text{ and a position } (u,v) \text{ such that } T_i(u,v) = p\} \\ h &:= \text{the maximum number of occurrences of } p \text{ in the column } j \text{ of some } T_i. \end{aligned}$$

By rearranging the T_i, we may assume that $T_1, T_2, \ldots, T_k$ $(k \leq s)$ are the tableaux which have the entry p exactly h times in their column j (necessarily in consecutive rows). We divide into three cases.

Case $p \in \{\overline{1}, \overline{2}, \ldots, \overline{r}\}$: Say $p = \overline{q}$ and let $I_n - \mu\, E_{q,j} + \mu\, E_{\overline{j},\overline{q}}$ act on $\sum_{i=1}^{s} c_i[T_i]$. The action of $I_n - \mu\, E_{q,j} + \mu\, E_{\overline{j},\overline{q}}$ on the generic matrix Y is the transformation $j \to j - \mu\, q$ and $p \to p + \mu\, \overline{j}$ applied to Y on both rows and columns. Since j never appears as an entry in all tableaux T_i, $1 \leq i \leq s$, by the symplectic standardness of each T_i and the minimality of p, only the transformation $p \to p + \mu\, \overline{j}$ matters.

Case $p = j$: Let $I_n + \mu\, E_{\overline{j},j}$ act on $\sum_{i=1}^{s} c_i[T_i]$. Note that the action of $I_n + \mu\, E_{\overline{j},j}$ on the generic matrix Y is the transformation $j \to j + \mu\, \overline{j}$ applied to Y on both rows and columns.

Case $p \in \{1, 2, \ldots, r\}$ with $p > j$: Let $I_n + \mu\, E_{\overline{j},p} + \mu\, E_{\overline{p},j}$ act on $\sum_{i=1}^{s} c_i[T_i]$. The action of $I_n + \mu\, E_{\overline{j},p} + \mu\, E_{\overline{p},j}$ on the generic matrix Y is the transformation $p \to p + \mu\, \overline{j}$ and $j \to j + \mu\, \overline{p}$ applied to Y on both rows and columns. Again j never appears as an entry in all tableaux T_i, so only the transformation $p \to p + \mu\, \overline{j}$ matters.

In each case, the action on $\sum_{i=1}^{s} c_i[T_i]$ results in a polynomial in μ of degree h, whose leading coefficient is $\sum_{i=1}^{k} c_i[T_i']$ where T_i' is obtained from $T_i, 1 \leq i \leq k$, by substituting $\overline{j}$ for the entries p in the column j. This polynomial has value 0 in $F[V]$ for all $\mu \in F$, so the leading coefficient must vanish by the following Lemma 4.10. It is not difficult to check that the tableaux $T_1', \ldots, T_k'$

are symplectic standard and distinct from each other. This new relation $\sum_{i=1}^{k} c_i[T_i'] = 0$ has either bigger p or same p with bigger j compared to the original relation $\sum_{i=1}^{s} c_i[T_i] = 0$. Hence, we are in an inductive procedure on (p, j) which ends with a relation in which only canonical tableaux are involved. It contradicts (1) so we get the desired conclusion. $\square$

Lemma 4.10. Assume F is an infinite field and let A be any F-algebra. If a polynomial $a_n x^n + \cdots + a_1 x + a_0 \in A[x]$ has more than n distinct roots in F, then a_n must be zero. (In fact, all a_i are then zero by iterated use of this lemma.)

Proof. We induct on the degree of the polynomial. Suppose $a_1 x + a_0 \in A[x]$ has two distinct roots $x_1, x_2 \in F$. It follows that

$$0 = (a_1 x_1 + a_0) - (a_1 x_2 + a_0) = a_1(x_1 - x_2).$$

Since $x_1 - x_2 \in F^\times$, clearly a_1 is equal to zero.

Now assume that the lemma holds true for all degree k polynomials in $A[x]$. Suppose that a polynomial $a_{k+1}x^{k+1} + \cdots + a_1 x + a_0 \in A[x]$ has more than $k+1$ distinct roots in F, and pick a root $\alpha \in F$. Then

$$\begin{aligned}
&a_{k+1}x^{k+1} + \cdots + a_2 x^2 + a_1 x + a_0 \\
&= a_{k+1}x^{k+1} + \cdots + a_2 x^2 + a_1 x + (-a_{k+1}\alpha^{k+1} - \cdots - a_2\alpha^2 - a_1\alpha) \\
&= a_{k+1}(x^{k+1} - \alpha^{k+1}) + \cdots + a_2(x^2 - \alpha^2) + a_1(x - \alpha) \\
&= (x - \alpha)(a_{k+1}\frac{x^{k+1} - \alpha^{k+1}}{x - \alpha} + \cdots + a_2\frac{x^2 - \alpha^2}{x - \alpha} + a_1).
\end{aligned}$$

Note that the second factor is a polynomial of degree k with leading coefficient a_{k+1}. For any root β of $a_{k+1}x^{k+1} + \cdots + a_1 x + a_0$ with $\beta \in F, \beta \neq \alpha$, evidently β must also be a root of the second factor. Hence this degree k polynomial has more than k distinct roots in F, and therefore $a_{k+1} = 0$ by the induction hypothesis. $\square$

Combining Proposition 4.7 and Proposition 4.9 (2), we come to the following conclusion:

Theorem 4.11. When F is a field with char $F = 0$ or char $F > r$, there is an F-basis for $F[V]$ indexed by the symplectic standard even-tableaux.

Corollary 4.12. Over any F-algebra R, there is an R-basis for $R[V]$ indexed by the symplectic standard even-tableaux.

Proof. It is an easy consequence of Theorem 4.11 and the base change from F to R. $\square$

Remark 4.13. When V is defined over a field F with char $F = 0$ or char $F > r$, the condition $\text{char}_{(-JY)}(T) = T^n$ imposed on V can be replaced by the weaker condition

$$\text{tr}(-JY) = 2\sum_{i=1}^{r} Y_{i\bar{i}} = 0. \tag{4.6}$$

Let V' be the scheme of $n \times n$ matrices Y defined by the conditions $Y = -Y^T$, $Y^T JY = 0$, and (4.6). In Section 3 and Section 4, we never used the condition $\text{char}_{(-JY)}(T) = T^n$ itself except for the weaker one (4.6). Hence if we start with V' instead of V, we can show that there is an F-basis for $F[V']$ indexed by the symplectic standard even-tableaux by the same argument. Then it is clear that $F[V'] = F[V]$.

When R is $\mathbb{Z}$ or $\mathbb{Q}$, let I_R denote the ideal of the polynomial ring $R[Y_{ij} : 1 \le i, j \le n]$ generated by the conditions (1.1), respectively. By definition, $R[V] = R[Y_{ij}]/I_R$. It is not difficult to see that a polynomial $f \in \mathbb{Z}[Y_{ij}]$ is in $I_{\mathbb{Q}}$ if and only if for some nonzero integer m, mf is in $I_{\mathbb{Z}}$. In other words, if we consider $\mathbb{Z}[V]$ as a $\mathbb{Z}$-module, $I_{\mathbb{Q}} \cap \mathbb{Z}[Y_{ij}]/I_{\mathbb{Z}}$ is the torsion submodule of $\mathbb{Z}[V]$, and the quotient module

$$S := \mathbb{Z}[V]/(I_{\mathbb{Q}} \cap \mathbb{Z}[Y_{ij}]/I_{\mathbb{Z}}) \cong \mathbb{Z}[Y_{ij}]/I_{\mathbb{Q}} \cap \mathbb{Z}[Y_{ij}]$$

is torsion-free. Note that the morphism $S \to \mathbb{Q}[V]$ (induced from $\mathbb{Z}[Y_{ij}] \to \mathbb{Q}[Y_{ij}]$) is injective, and hence we can regard S as a subring of $\mathbb{Q}[V]$.

Proposition 4.14. There is a $\mathbb{Z}$-basis for S indexed by the symplectic standard even-tableaux.

Proof. It suffices to show that S is the $\mathbb{Z}$-span of the set

$$\{[T] \in \mathbb{Q}[V] : T \text{ is a symplectic standard even-tableau}\}$$

in $\mathbb{Q}[V]$. Pick any nonzero $f \in S$. By Theorem 4.11, we can write f as

$$f = \sum_{i=1}^{k} c_i [T_i],$$

where the T_i are symplectic standard even-tableaux and coefficients c_i are taken in $\mathbb{Q}$. We show that c_i are actually in $\mathbb{Z}$. Express each c_i in a reduced form $c_i = a_i/b_i$ with $a_i, b_i \in \mathbb{Z}, b_i \neq 0$, and let l be the least common multiple of $b_1, b_2, \cdots, b_k$. Multiplying l to both sides of the equation, we get an expression with coefficients in $\mathbb{Z}$:

$$lf = \sum_{i=1}^{k} (lc_i)[T_i].$$

Suppose $l > 1$ and pick a prime factor p of l. Modulo p we get a nontrivial relation, which contradicts Proposition 4.9 (2). Hence l must be equal to 1, and this proves the proposition. $\square$

When $F = \mathbb{C}$, there is an alternative way to prove Proposition 4.9 (2). For any even-tableau T, the element $[T]$ is homogeneous in the graded ring $\mathbb{C}[V]$. Thus, it suffices to prove that the map from the set

$$\{T \mid T \text{ is a symplectic standard even-tableau whose shape is a partition of } 2m\} \tag{4.7}$$

into $F[V]_m$, given by $T \mapsto [T]$, is injective and its image in $F[V]_m$ is linearly independent over F. Given a partition λ (shape of a tableau), we use the following notations:

row$(i, \lambda) :=$ the size of row i of λ and possibly equal to zero if λ does not have row i,

col$(j, \lambda) :=$ the size of column j of λ and possibly equal to zero if λ does not have column j.

Proposition 4.15. Let $\mathcal{P}_m$ be the set of all partitions λ of 2m whose row(i, λ) is even and $\leq r$ for all i. The degree m homogeneous part of $\mathbb{C}[V]$ can be decomposed as

$$\mathbb{C}[V]_m = \bigoplus_{\lambda \in \mathcal{P}_m} L_\lambda E,$$

where $L_\lambda E$ denotes the irreducible representation of $Sp(n, \mathbb{C})$ whose highest weight is

$$col(1,\lambda), col(2,\lambda), \ldots, col(n,\lambda).$$

Proof. We first show the inclusion

$$\mathbb{C}[V]_m \supseteq \bigoplus_{\lambda \in \mathcal{P}_m} L_\lambda E,$$

and the proof is similar to that of [14, Prop.2.3.8 (b)]. For each even number s, $0 \leq s \leq r$, we define $g_s := [1, 2, \ldots, s] \in \mathbb{C}[V]$. Let U be the subgroup of $Sp(n, \mathbb{C})$ consisting of all upper triangular matrices with 1's on the diagonal. It is immediate that for each s, g_s is a U-invariant of the weight $(1^s, 0^{n-s})$. Now for any partition $\lambda \in \mathcal{P}_m$, we see that $g_\lambda := \prod_i g_{row(i,\lambda)}$ is a nonzero U-invariant of the weight

$$col(1,\lambda), col(2,\lambda), \ldots, col(n,\lambda),$$

and therefore $\mathbb{C}[V]_m \supseteq L_\lambda E$. (By a slight modification of the proof of Proposition 4.9(1), we can show that g_λ is nonzero in $\mathbb{C}[V]$.)

From [5], we know that

$$dim_{\mathbb{C}}(L_\lambda E) = \text{the number of symplectic standard even-tableaux of shape } \lambda$$

for every $\lambda \in \mathcal{P}_m$. Note that if T is in (4.7), then the shape of T is in $\mathcal{P}_m$. By the proof of Proposition 4.7, the set (4.3) spans $\mathbb{C}[V]_m$. It follows that

$$dim_{\mathbb{C}}(\mathbb{C}[V]_m) \leq \sum_{\lambda \in \mathcal{P}_m} dim_{\mathbb{C}}(L_\lambda E),$$

and therefore the equality

$$\mathbb{C}[V]_m = \bigoplus_{\lambda \in \mathcal{P}_m} L_\lambda E.$$

This also proves that the set (4.3) is a $\mathbb{C}$-basis of $\mathbb{C}[V]_m$. $\square$

5 Reducedness of the coordinate ring

We assume that $n = 2r$ is a multiple of 4 and that F is a field with char $F = 0$ or char $F > r$. Then Remark 4.13 implies

$$F[V] = F[Y_{ij} : 1 \leq i, j \leq n]/(Y^T JY,\, Y + Y^T,\, \sum_{i=1}^{r} Y_{i\bar{i}}),$$

where $Y = (Y_{ij})$ denotes $n \times n$ generic matrix. We define f as the minor of Y consisting of the rows $\bar{1}, \bar{2}, \ldots, \bar{r}$ and columns $\bar{1}, \bar{2}, \ldots, \bar{r}$.

Lemma 5.1. The element f is not a zero divisor in $F[V]$.

Proof. This follows the proof of [7, Cor.4.4]. It suffices to show that the pfaffian $[\bar{1}, \bar{2}, \ldots, \bar{r}]$ is not a zero divisor in $F[V]$ since f is the square of $[\bar{1}, \bar{2}, \ldots, \bar{r}]$. ($[\bar{1}, \bar{2}, \ldots, \bar{r}] = 0$ if r is odd, so we need the assumption that r is even.) The one row tableau associated to $[\bar{1}, \bar{2}, \ldots, \bar{r}]$ is obviously symplectic standard. Furthermore, for any symplectic standard even-tableau T, the product $[\bar{1}, \bar{2}, \ldots, \bar{r}] \cdot [T]$ is again associated to a symplectic standard even-tableau. Since the elements of $F[V]$ indexed by the symplectic standard even-tableaux form a basis of $F[V]$, this clearly implies that $[\bar{1}, \bar{2}, \ldots, \bar{r}]$ is not a zero divisor. $\square$

Lemma 5.2. The scheme Spec $F[V]_f$ is isomorphic to an open subscheme of $\mathbb{A}^{r^2}$.

Proof. Consider two affine open subschemes of the Grassmannian scheme $Gr(n, 2n)$ where we can represent the n-dimensional subspaces as

$$M = \left[\begin{array}{ccc} Y_{1,1} & \cdots & Y_{1,n} \\ \vdots & & \vdots \\ Y_{n,1} & \cdots & Y_{n,n} \\ \hline 1 & & \\ & \ddots & \\ & & 1 \end{array}\right], \qquad N = \left[\begin{array}{ccc} Z_{1,1} & \cdots & Z_{1,n} \\ \vdots & & \vdots \\ Z_{r,1} & \cdots & Z_{r,n} \\ \hline 1 & & \\ & \ddots & \\ & & 1 \\ \hline Z_{r+1,1} & \cdots & Z_{r+1,n} \\ \vdots & & \vdots \\ Z_{n,1} & \cdots & Z_{n,n} \end{array}\right].$$

Let H be the $n \times n$ submatrix of M given by

$$H = \left[\begin{array}{cccccc} Y_{r+1,1} & \cdots & Y_{r+1,r} & Y_{r+1,r+1} & \cdots & Y_{r+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ Y_{n,1} & \cdots & Y_{n,r} & Y_{n,r+1} & \cdots & Y_{n,n} \\ \hline 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \end{array}\right],$$

and G be the $n \times n$ submatrix of N defined as

$$G = \left[\begin{array}{cccccc} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ \hline Z_{r+1,1} & \cdots & Z_{r+1,r} & Z_{r+1,r+1} & \cdots & Z_{r+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ Z_{n,1} & \cdots & Z_{n,r} & Z_{n,r+1} & \cdots & Z_{n,n} \end{array}\right].$$

Let h and g denote the determinants of H and G respectively. We have a ring isomorphism between localizations $F[Y_{i,j}]_h$ and $F[Z_{i,j}]_g$ matching $Y_{i,j}$ to the entry (i, j) of the following product of matrices:

$$\left[\begin{array}{cccccc} Z_{1,1} & \cdots & Z_{1,r} & Z_{1,r+1} & \cdots & Z_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ Z_{r,1} & \cdots & Z_{r,r} & Z_{r,r+1} & \cdots & Z_{r,n} \\ \hline 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \end{array}\right] \left[\begin{array}{cccccc} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ \hline Z_{r+1,1} & \cdots & Z_{r+1,r} & Z_{r+1,r+1} & \cdots & Z_{r+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ Z_{n,1} & \cdots & Z_{n,r} & Z_{n,r+1} & \cdots & Z_{n,n} \end{array}\right]^{-1}.$$

(The morphism $F[Y_{i,j}]_h \xrightarrow{\sim} F[Z_{i,j}]_g$ is a glueing datum of the affine open subschemes of $Gr(n, 2n)$.)

Here I_n denotes $n \times n$ identity matrix and O_n denotes $n \times n$ zero matrix. Consider the following conditions given to a $2n \times n$ matrix U:

(1) $\left(\left[\begin{array}{c|c} I_n & O_n \end{array}\right] U\right)^T J \left(\left[\begin{array}{c|c} I_n & O_n \end{array}\right] U\right) = O_n$,

(2) $U^T \left[\begin{array}{c|c} O_n & I_n \\ \hline I_n & O_n \end{array}\right] U = O_n$.

Case $U = M$: the conditions are equivalent to

(1) $Y^T J\, Y = O_n$,

(2) $Y + Y^T = O_n$,

where $Y = (Y_{i,j})$ denotes the $n \times n$ submatrix of M. Let α be the ideal of $F[Y_{i,j}]$ generated by these conditions on Y.

Case $U = N$: for convenience, we divide the $n \times n$ matrix $(Z_{i,j})$ into four $r \times r$ blocks

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then equivalent conditions are

(1) $A^T = A, \quad B = O_r$,

(2) $C^T = -C, \quad B^T = -B, \quad D = -A^T$,

so (1) and (2) together:

$$A^T = A, \quad B = O_r, \quad C^T = -C, \quad D = -A^T. \tag{5.1}$$

When β is the ideal of $F[Z_{i,j}]$ generated by (5.1), it is easy to see that $F[Z_{i,j}]/\beta$ is isomorphic to the polynomial ring in indeterminates $\{A_{i,j}\}_{i \le j}$ and $\{C_{i,j}\}_{i<j}$. That is, $\mathrm{Spec}\big(F[Z_{i,j}]/\beta\big) \cong \mathbb{A}^{r^2}$.

We claim that the morphism $F[Y_{i,j}]_h \xrightarrow{\sim} F[Z_{i,j}]_g$ induces an isomorphism between quotients

$$(F[Y_{i,j}]/\alpha)_h \quad \cong \quad F[Y_{i,j}]_h/\alpha_h \quad \to \quad F[Z_{i,j}]_g/\beta_g \quad \cong \quad (F[Z_{i,j}]/\beta)_g. \tag{5.2}$$

Generators of α from the condition $Y^T J Y = O_n$ are entries of the product of matrices

$$(\left[\begin{array}{c|c} I_n & O_n \end{array}\right] M)^T J \,(\left[\begin{array}{c|c} I_n & O_n \end{array}\right] M),$$

and under the map $F[Y_{i,j}]_h \to F[Z_{i,j}]_g$ each entry maps to the entry at the exact same location of the product of matrices

$$(\left[\begin{array}{c|c} I_n & O_n \end{array}\right] NG^{-1})^T J \,(\left[\begin{array}{c|c} I_n & O_n \end{array}\right] NG^{-1}),$$

which is equal to

$$(G^{-1})^T (\left[\begin{array}{c|c} I_n & O_n \end{array}\right] N)^T J \,(\left[\begin{array}{c|c} I_n & O_n \end{array}\right] N) G^{-1}.$$

Note that entries of the product

$$(\left[\begin{array}{c|c} I_n & O_n \end{array}\right] N)^T J \,(\left[\begin{array}{c|c} I_n & O_n \end{array}\right] N)$$

are in β. It follows that generators of α from the condition $Y^T J\, Y = O_n$ map to elements of β_g. Likewise, the generators of α obtained from $Y + Y^T = O_n$ are entries of the product

$$M^T \left[\begin{array}{c|c} O_n & I_n \\ \hline I_n & O_n \end{array}\right] M,$$

which map to the entries at the same location of

$$(NG^{-1})^T \left[\begin{array}{c|c} O_n & I_n \\ \hline I_n & O_n \end{array}\right] NG^{-1} \quad = \quad (G^{-1})^T \{N^T \left[\begin{array}{c|c} O_n & I_n \\ \hline I_n & O_n \end{array}\right] N\} G^{-1}.$$

These entries are in β_g, so we know that the morphism (5.2) is well-defined. In the same way, we can show that generators of β map to elements of α_h under the inverse morphism $F[Z_{i,j}]_g \to F[Y_{i,j}]_h$. This proves that the morphism (5.2) is isomorphic.

Finally, we consider the last condition $\sum_{i=1}^r Y_{i\bar{i}} = 0$ imposed on V. We are interested in the image of $\sum_{i=1}^r Y_{i\bar{i}}$ under the morphism (5.2). It is not difficult to see that

$$\sum_{i=1}^{r} Y_{i\bar{i}} = \text{tr}\left(\begin{bmatrix} Y_{1,\bar{1}} & \cdots & Y_{1,\bar{r}} \\ \vdots & & \vdots \\ Y_{r,\bar{1}} & \cdots & Y_{r,\bar{r}} \end{bmatrix}\right)$$

maps to tr(AC^{-1}) under the morphism $F[Y_{i,j}]_h \xrightarrow{\sim} F[Z_{i,j}]_g$. Let $C^{i,j}$ denote the cofactor of the (i,j) entry of the matrix C. Then

$$\begin{aligned}\text{tr}(AC^{-1}) &= \frac{1}{det(C)} \sum_{i=1}^{r} \sum_{j=1}^{r} A_{i,j} C^{i,j} \\ &= \frac{1}{det(C)} \{\sum_{i=1}^{r} A_{i,i} C^{i,i} + \sum_{i<j} (A_{i,j} C^{i,j} + A_{j,i} C^{j,i})\}.\end{aligned}$$

Recall that r is an even number. Since C is a skew-symmetric $r \times r$ matrix, $C^{i,j} = -C^{j,i}$ if $i < j$ and $C^{i,i} = 0$ for all i. Furthermore the matrix A is symmetric, so we have tr$(AC^{-1}) = 0$ in $F[Z_{i,j}]_g/\beta_g$. In conclusion,

$$F[V]_f \cong \big(F[Y_{i,j}]/\alpha + (\sum_{i=1}^{r} Y_{i\bar{i}})\big)_h \cong (F[Z_{i,j}]/\beta)_g. \tag{5.3}$$

Since Spec $(F[Z_{i,j}]/\beta) \cong \mathbb{A}^{r^2}$, it proves the lemma. □

Theorem 5.3. The coordinate ring $F[V]$ is an integral domain.

Proof. From the isomorphisms (5.3), we know that $F[V]_f$ is an integral domain. Then by Lemma 5.1, there is an embedding $F[V] \to F[V]_f$, and hence $F[V]$ must also be an integral domain. □

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